FIITJEE

L I M I T E D

Regional Mathematical Olympiad-2018

Time : 3 hours **October 07, 2018**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks: 102
- Answers to each question should start on a new page. Clearly indicate the question number.

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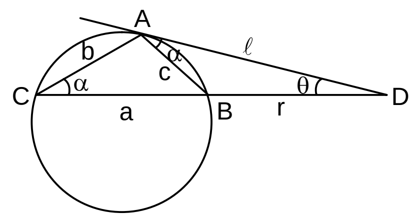
1. Let ABC be a triangle with integer sides in which AB < AC. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D. Suppose AD is also an integer. Prove that gcd(AB, AC) > 1.



Sol. Here Δ DCA and ΔDAB are similar

So $\frac{c}{b} = \frac{r}{\ell} = \frac{\ell}{a+r}$

$r = \frac{\ell c}{b}$, $c(a + r) = \ell b$

$\Rightarrow c\left(a + \frac{\ell c}{b}\right) = \ell b$

$abc + \ell c^2 = \ell b^2$

$abc = \left(b^2 - c^2\right)\ell$

Let gcd of (b, c) = 1 so b will not divide $b^2 - c^2$ and also c will not divide $b^2 - c^2$

i.e. $\ell = \lambda bc$

$a = (b^2 - c^2)\lambda$ and given that b + c > a

$b + c > (b^2 - c^2)\lambda$

$(b - c)\lambda < 1$ which is not possible

So, gcd = b, c will be > 1

2. Let n be a natural number. Find all real numbers x satisfying the equation

$$\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

Sol. $\frac{1}{x+\frac{1}{x}} + \frac{2}{x^2+\frac{1}{x^2}} + \frac{3}{x^3+\frac{1}{x^3}} + ... + \frac{n}{x^n+\frac{1}{x^n}} = \frac{n(n+1)}{4}$

$x + \frac{1}{x} \ge 2 \ (x > 0) \Rightarrow \frac{1}{x+\frac{1}{x}} \le \frac{1}{2}$

$\text{L.H.S.} \le \frac{1}{2} + \frac{2}{2} + \frac{3}{2} + ... + \frac{n}{2} \le \frac{n(n+1)}{4}$

$\Rightarrow$ Equality holds for x = 1 and when x < 0, L.H.S. < $\frac{n(n+1)}{4}$

3. For a rational number r, its period is the length of the smallest repeating block in its decimal expansion. For example, the number r = $0.\overline{123}123123...$ has period 3. If S denotes the set of all rational numbers r of the form $r = 0.\overline{abcdefgh}$ having period 8, find the sum of all the elements of S.

Sol. Sum of numbers with period 1 is $\frac{1+2+3+.....+8}{9} = 4$

Sum of numbers with period 2 is $\frac{1+2+3+.....+98}{99} - 4 = 45$

Sum of numbers with period 4 is $\frac{1+2+3+.....+9998}{9999} - 4 - 45 = 4950$

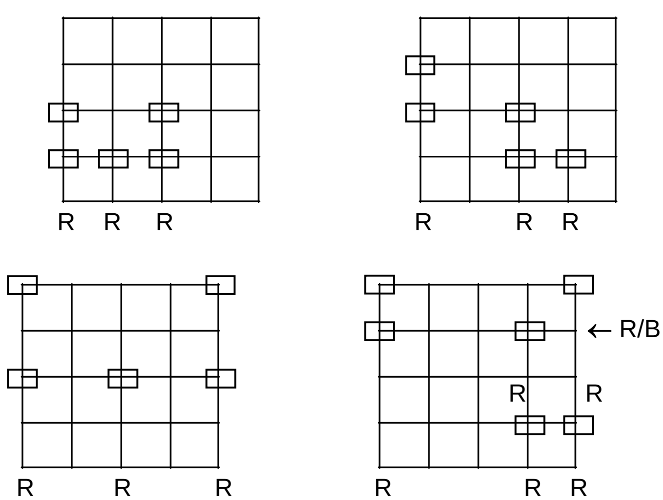
Sum of numbers with period 8 is $\frac{1+2+3+.....+99999998}{99999999} - 4 - 45 - 4950 = 49995000$

4. Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \le m \le 5$, $1 \le n \le 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form (a, b), (a + k, b), (a + k, b + k), (a, b + k) for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square with sides parallel to axes and having at least three points of the same colour).

Sol. Every row has atleast 3 vertices of same colour. Without loss of generality we can assume this to be red for first row, so 4 possibilities arises

In each figure any vertex denoted by □ can not be of red colour otherwise there will be a square whose atleast three vertices will be of red colour. If all will be of blue colour then there will be a square whose atleast three vertices will be blue

In fourth case the vertex which is indicated by the arrow will be either red or blue. Again we will get a square whose three vertices are of same colour



Alternate solution:

If we try to create 5 × 5 grid such that no 'unit' square exists such that if has atleast three points of same colour, then we will have to colour them alternatively along atleast one of the side of square grid. In doing so, it is obvious that a square of side 2 units will be formed which has atleast 3 points of same colour as (a, b) will have same colour as (a + 2, b) and (a, b + 2).

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5. Find all natural numbers n such that $1+\left[\sqrt{2n}\right]$ divides 2n. (For any real number x, [x] denotes the largest integer not exceeding x).

Sol. **Case-I:** 2n is not a perfect square then $\sqrt{2n} < 1+\left[\sqrt{2n}\right] < \sqrt{2n}+1$

Let $1+\left[\sqrt{2n}\right] = x$

So, $\sqrt{2n} < x < \sqrt{2n}+1$

$(x-1)^2 < 2n < x^2$

But x | 2n

$\therefore$ 2n must be $x^2 - x$

$\therefore x^2 - 2x + 1 < 2n < x^2$

$n = \frac{x(x-1)}{2} \ \forall\, x \geq 2$

Case-II: 2n is a perfect square

Let $2n = a^2$ for some $a \in N$

$\left[\sqrt{2n}\right] + 1 = a + 1$

$\therefore$ We want $a + 1 \mid a^2$ but gcd (a + 1, a) = 1 and $a \neq 0$

gcd $(a + 1, a^2) = 1$

So, a + 1 does not divide a^2

$\therefore\ n = \frac{x(x-1)}{2} \ \forall\, x \geq 2$ and $x \in I$ is the only solution

6. Let ABC be an acute-angled triangle with AB < AC. Let I be the incentre of triangle ABC, and let D, E, F be the points at which its incircle touches the sides BC, CA, AB, respectively. Let BI, CI meet the line EF at Y, X, respectively. Further assume that both X and Y are outside the triangle ABC. Prove that

(i) B, C, Y, X are concyclic; and

(ii) I is also the incentre of triangle DY X.

Sol. **(i)** $\angle BIC = \frac{\pi}{2} + \frac{A}{2}$

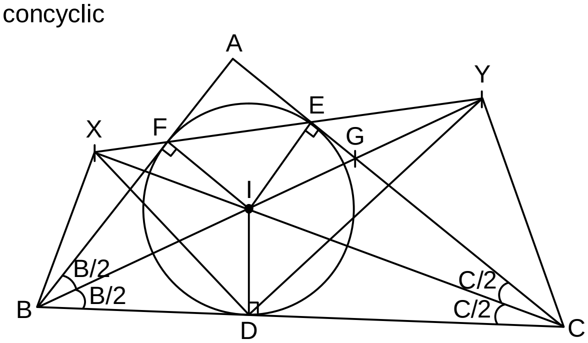
$\angle IGC = \frac{\pi}{2} + \frac{A}{2} - \frac{C}{2} = \angle EGY$

$\angle FEA = \frac{\pi}{2} - \frac{A}{2} = \angle GEY$

$\Rightarrow \angle EYG = \pi - \left(\frac{\pi}{2} + \frac{A}{2} - \frac{C}{2} + \frac{\pi}{2} - \frac{A}{2}\right) = \frac{C}{2}$

$\Rightarrow \angle EYG = \angle XYB = \frac{C}{2} = \angle XCB$

$\Rightarrow$ BCYX is concyclic



(ii) Also BY = BI + IY = $DI\,\text{cosec}\left(\frac{B}{2}\right) + \frac{IE\sin\left(\frac{A}{2}\right)}{\sin\left(\frac{C}{2}\right)}$

$\Rightarrow\ BY = DI\left[\frac{\cos\left(\frac{B}{2}\right)\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{C}{2}\right)\sin\left(\frac{B}{2}\right)}\right], (\text{as } DI = IE)$

$\Rightarrow\ BY = BC\cos\left(\frac{B}{2}\right)$

$\Rightarrow\ \angle BYC = \frac{\pi}{2} = \angle IDC$

$\Rightarrow$ IDCY is concyclic

$\Rightarrow \angle IYD = \angle ICD = \frac{C}{2}$ similarly $\angle IXD = \angle IBD = \frac{B}{2}$

$\Rightarrow$ I is in-centre of ΔDXY.

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